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A three-dimensional model of the wind-driven ocean circulation

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A linear three-dimensional model of the wind-driven ocean circulation is treated by boundary-layer methods. The interior flow, below the Ekman layer, differs from the classical gyres of Munk (1950). There is a north-eastwards transport of fluid from the western boundary current of the southern gyre across the latitude of zero wind stress curl into the northern gyre. A return flow in the Ekman layer preserves continuity.

1. Introduction

The standard method for determining the ocean currents produced by a given distribution of wind stress involves integrating all variables over a predetermined depth of the ocean and thenceforth working with volume transports in place of velocity components. This technique overcomes the difficulties inherent in dealing with a three-dimensional problem and with variable density. However, no insight is given into the actual variations with depth of the velocity and density fields, and the difficulties surrounding the choice of the level of no motion have been discussed by Stommel (1958, chapter 3). The results obtained by this method consist of discrete gyres, whereas from examination of charts of mean ocean currents it is seen that the circulation is essentially three-dimensional with fluid flowing from one gyre to another, returning at some other depth. This return flow is usually assumed to be part of the thermohaline circulation.

Stommel (1957) suggested that the circulation in a three-dimensional model has the same gyre structure as the vertically integrated transport model. In this paper, by considering a very simplified three-dimensional model, it is shown that the process of vertically integrating masks certain interesting features of the ocean circulation. In particular, we show that fluid can be transported between gyres and a return flow take place within the framework of a homogeneous ocean. Thus, in addition to the thermohaline circulation, a second mechanism exists for dealing with different velocity fields at different depths.

2. Formulation

The model under consideration is a rectangular homogeneous ocean of uniform depth D on a β-plane, with the Cartesian co-ordinate system shown in figure 1 that has x increasing eastwards, y northwards and z vertically upwards. The corresponding velocity components are $\mathbf{u} = (u, v, w)$ and the vertical component of the Coriolis parameter $f = f_0 + \beta y$ is the familiar form used in the β-plane approximation. If P is the reduced pressure, ν_H and ν_V are respectively the horizontal and vertical coefficients of eddy viscosity, the momentum equation for steady flow is

$$\mathbf{u}.\nabla\mathbf{u} + f\mathbf{k}\times\mathbf{u} = -\nabla P + \nu_H\nabla^2\mathbf{u} + \nu_V\frac{\partial^2\mathbf{u}}{\partial z^2}, \tag{1}$$

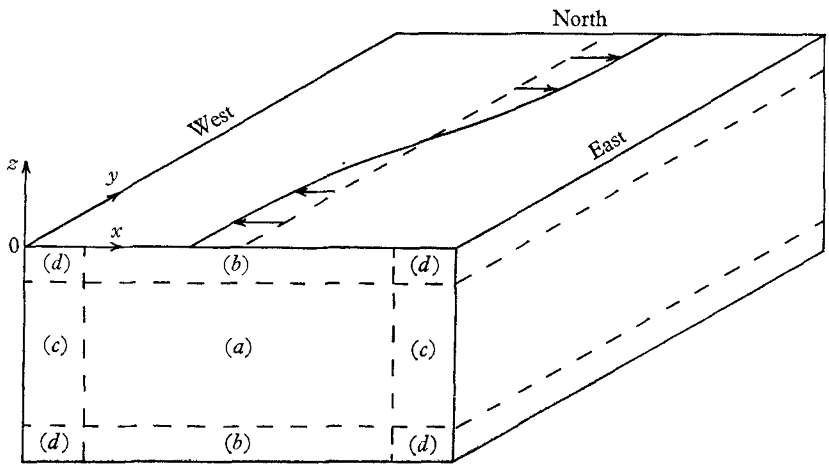

FIGURE 1. Notation.

where $\mathbf{k} = (0,0,1)$ and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The assumption that ν_V is constant is reasonable except near the bottom, but we shall see that the Ekman layer at the bottom is unimportant anyway. Equation (1) includes the fact that the dominant Coriolis accelerations are produced by the vertical component of the Earth's angular velocity.

The system may be made non-dimensional by introducing the following new variables

$$(x,y) = L(x',y'),\quad z = \epsilon L z',\quad (u,v) = U(u',v'),\quad w = \epsilon U w',\quad P = Uf_0LP',$$

where L is the east–west dimension of the ocean, U is a velocity scale related to the amplitude of the imposed surface stress, and $\epsilon = D/L$ is a small parameter indicating the shallowness of the ocean. In non-dimensional form the components of (1) become

$$R\mathbf{u}.\nabla u - (1+\beta^* y)v = -\frac{\partial P}{\partial x} + E_H\nabla^2 u + E_V\frac{\partial^2 u}{\partial z^2}, \tag{2}$$

$$R\mathbf{u}.\nabla v + (1+\beta^* y)u = -\frac{\partial P}{\partial x} + E_H\nabla^2 v + E_V\frac{\partial^2 v}{\partial z^2}, \tag{3}$$

$$R\mathbf{u}.\nabla w = -\frac{1}{\epsilon^2}\frac{\partial P}{\partial z} + E_H\nabla^2 w + E_V\frac{\partial^2 w}{\partial z^2}, \tag{4}$$

where the primes have been dropped, the parameters R, β^*, E_H, E_V are defined as

$$R = U/f_0L,\quad \beta^* = \beta L/f_0,\quad E_H = \nu_H/f_0L^2,\quad E_V = \nu_V/f_0D^2,$$

and the Coriolis parameter now has the form $f = f_0(1+\beta^* y)$. The system is closed by the continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0. \tag{5}$$

The boundary conditions in non-dimensional form are as follows. The conditions at the surface $z = 0$ are

$$\tau_x = \partial u/\partial z = -E_V^{-\frac{1}{2}}\cos \pi y, \tag{6a}$$

$$\tau_y = \partial v/\partial z = 0, \quad w = 0, \tag{6b}$$

where τ_x, τ_y are the components of wind stress in the x, y directions respectively. At the rigid boundaries at the bottom and on the east and west coasts, the velocity is zero,

$$u = v = w = 0; \quad z = -1, \quad x = 0, 1.$$

The north and south boundaries remain free, but we are particularly interested in the region between $y = 0$ and $y = 2$, latitudes at which the wind stress curl is zero.

3. Numerical values

Five parameters ϵ, R, β^*, E_H, E_V have been introduced in the previous section and before further approximation can be made to the equations of motion, it is necessary to decide the relative magnitudes of their terms. The following are taken as typical values for the dimensional parameters:

$$f_0 = 7{\cdot}3\times 10^{-5}\,\text{sec}^{-1}, \quad \beta = 2{\cdot}6\times 10^{-13}\,\text{cm}^{-1}\,\text{sec}^{-1},$$

$$L = 2\times 10^{8}\,\text{cm}, \quad D = 4\times 10^{5}\,\text{cm}, \quad U = 5\,\text{cm}\,\text{sec}^{-1},$$

$$\nu_H = 7\times 10^{6}\,\text{cm}^2\,\text{sec}^{-1}, \quad \nu_V = 10^{2}\,\text{cm}^2\,\text{sec}^{-1}.$$

The Coriolis parameters are evaluated at latitude 15° N, and for the eddy viscosities we have used a value for ν_H that was given by Arons & Stommel (1967) and for ν_V a value that is appropriate near the surface.

Using these values, the Rossby number

$$R = 3{\cdot}4\times 10^{-4}.$$

Hence the non-linear terms will be small compared with the Coriolis terms except perhaps in narrow western boundary currents where velocities are $O(100)$ cm sec^{-1}. However, in this paper we shall confine our attention to viscous boundary layers, neglecting the effects due to the non-linear terms. The Ekman numbers are

$$E_H = 2{\cdot}4\times 10^{-6}, \quad E_V = 8{\cdot}5\times 10^{-6}.$$

As these values for E_H and E_V are close, we shall assume that $E_H/E_V \sim O(1)$ when using power series expansions in terms of the Ekman numbers. Slight variations in the choice of eddy viscosities will only modify the solution quantitatively. The remaining non-dimensional parameters are

$$\epsilon = 2\times 10^{-3}, \quad \beta^* = 0{\cdot}71.$$

The latter cannot reasonably be neglected compared with unity, and it is the inclusion of this term throughout that gives rise to the interesting results. To summarize, the assumptions to be made are

$$R \sim 0; \quad E_H/E_V,\ \beta^* \sim O(1); \quad \epsilon,\ E_H,\ E_V \ll 1.$$

4. The vorticity equation

Neglecting the non-linear terms, cross-differentiation of (2)–(4) and use of the continuity equation (5) leads to the vorticity equations

$$-(1+\beta^* y)\frac{\partial w}{\partial z}+\beta^* v = \left(E_H\nabla^2+E_V\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right), \tag{7}$$

$$-(1+\beta^* y)\frac{\partial u}{\partial z} = \left(E_H\nabla^2+E_V\frac{\partial^2}{\partial z^2}\right)\left(\epsilon^2\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right), \tag{8}$$

$$-(1+\beta^* y)\frac{\partial v}{\partial z} = \left(E_H\nabla^2+E_V\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial u}{\partial z}-\epsilon^2\frac{\partial w}{\partial x}\right). \tag{9}$$

The method used to deal with these equations will be power series expansion for the dependent variables, each expansion valid in one of the regions shown in figure 1. These regions are: (*a*) the interior away from the frictional influence of the boundaries at $x = 0,\ 1$; $z = 0,\ -1$; (*b*) the Ekman layers of thickness $O(E_V^{\frac{1}{2}})$ near $z = 0,\ -1$ but excluding the regions $O(E_V^{\frac{1}{2}})$ from the coasts at $x = 0, 1$; (*c*) the east and west coast layers of thickness $O(E_H^{\frac{1}{3}})$ but excluding the regions $O(E_V^{\frac{1}{2}})$ from the boundaries at $z = 0,\ -1$; (*d*) the corner regions along the east and west coasts but within a distance $O(E_V^{\frac{1}{2}})$ from the boundaries at $z = 0,\ -1$. This region is only discussed in terms of continuity.

5. The interior

In this region, away from the frictional influence of the boundaries, the viscous terms in the vorticity equations will be small. Expanding formally, we write

$$u = u_0+E_V^{\frac{1}{6}}u_1+E_V^{\frac{1}{3}}u_2+\ldots,$$
$$v = v_0+E_V^{\frac{1}{6}}v_1+E_V^{\frac{1}{3}}v_2+\ldots,$$
$$w = w_0+E_V^{\frac{1}{6}}w_1+E_V^{\frac{1}{3}}w_2+\ldots,$$

where we have chosen the small parameter as $E_V^{\frac{1}{6}}$ for conformity in the series for all regions. Substitution into the vorticity equations (7)–(9) yields for $i = 1$ to 5,

$$-(1+\beta^* y)\frac{\partial w_i}{\partial z}+\beta^* v_i = 0, \tag{10a}$$

$$\partial u_i/\partial z = 0, \quad \partial v_i/\partial z = 0. \tag{10b}$$

For higher orders, a contribution from the viscous terms appears. The continuity equation must always be satisfied and is, for all i,

$$\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}+\frac{\partial w_i}{\partial z} = 0. \tag{11}$$

The solution of (10) is, for $i = 1$–5,

$$u_i = u_i(x,y), \quad v_i = v_i(x,y), \quad (1+\beta^*y)w_i = \beta^* v_i z + w_i^*(x,y), \tag{12}$$

where the unknown functions must be found by matching with the flow in the Ekman layers and the side-wall layers. The horizontal components are independent of depth, whereas the vertical velocity varies linearly with depth.

6. The Ekman layers

The upper Ekman layer

In this thin layer vertical gradients of the velocity components will be large. We introduce a magnified vertical co-ordinate $\bar{\zeta} = z/(2E_V)^{\frac{1}{2}}$ and let $\bar{w} = (2E_V)^{\frac{1}{2}}\overline{W}$. The overbars will indicate variables appropriate to the upper Ekman layer and we write

$$\bar{u} = \bar{u}_0 + E_V^{\frac{1}{2}}\bar{u}_1 + \dots,$$

$$\bar{v} = \bar{v}_0 + E_V^{\frac{1}{2}}\bar{v}_1 + \dots,$$

$$\bar{w} = (2E_V)^{\frac{1}{2}}\overline{W} = (2E_V)^{\frac{1}{2}}\{\overline{W}_0 + E_V^{\frac{1}{2}}\overline{W}_1 + \dots\}.$$

On substituting into (7)–(9) we have, for $i = 1$–5,

$$(1+\beta^*y)\overline{W}_{i\bar{\zeta}} - \beta^*\bar{v}_i + \tfrac{1}{2}(\bar{v}_{ix\bar{\zeta}\bar{\zeta}} - \bar{u}_{iy\bar{\zeta}\bar{\zeta}}) = 0, \tag{13}$$

$$(1+\beta^*y)\bar{u}_{i\bar{\zeta}} - \tfrac{1}{2}\bar{v}_{i\bar{\zeta}\bar{\zeta}\bar{\zeta}} = 0, \tag{14}$$

$$(1+\beta^*y)\bar{v}_{i\bar{\zeta}} + \tfrac{1}{2}\bar{u}_{i\bar{\zeta}\bar{\zeta}\bar{\zeta}} = 0, \tag{15}$$

and the continuity equation becomes

$$\bar{u}_{ix} + \bar{v}_{iy} + \overline{W}_{i\zeta} = 0. \tag{16}$$

The suffices denote differentiation, and either (13) or (16) may be used to determine $\overline{W}_i$ as the four equations are self-consistent.

The zeroth-order solution may be calculated by combining (14) and (15), which are similar to the normal Ekman layer equations, to give

$$\frac{\partial^5}{\partial\bar{\zeta}^5}(\bar{u}_0, \bar{v}_0) + 4(1+\beta^*y)^2\frac{\partial}{\partial\bar{\zeta}}(\bar{u}_0, \bar{v}_0) = 0,$$

which has the following solution bounded as $\bar{\zeta} \to -\infty$,

$$\bar{u}_0 = \bar{u}_{01}(x,y) + \bar{u}_{02}(x,y)\exp[(1+i)B\bar{\zeta}] + \bar{u}_{03}(x,y)\exp[(1-i)B\bar{\zeta}],$$

$$\bar{v}_0 = \bar{v}_{01}(x,y) + \bar{v}_{02}(x,y)\exp[(1+i)B\bar{\zeta}] + \bar{v}_{03}(x,y)\exp[(1-i)B\bar{\zeta}],$$

where $B^2 = 1+\beta^*y$. Matching with the interior flow (12) for large negative $\bar{\zeta}$ gives

$$\bar{u}_{01} = u_0, \quad \bar{v}_{01} = v_0.$$

The boundary conditions (6) at $z = 0$ in terms of the new magnified variable $\bar{\zeta}$ are

$$\partial\bar{u}_0/\partial\bar{\zeta} = -\cos\pi y, \quad \partial\bar{v}_0/\partial\bar{\zeta} = 0, \quad \overline{W}_0 = 0, \quad \bar{\zeta} = 0, \tag{17}$$

and they yield the conditions

$$(1+i)B\bar{u}_{02} + (1-i)B\bar{u}_{03} = -\cos\pi y,$$

$$(1+i)B\bar{v}_{02} + (1-i)B\bar{v}_{03} = 0.$$

Now substitution of the solutions into (14) together with these boundary conditions gives

$$\bar{u}_{02} = i\bar{v}_{02} = -\frac{1-i}{4B}\cos \pi y, \quad \bar{u}_{03} = -i\bar{v}_{03} = -\frac{1+i}{4B}\cos \pi y.$$

Thus the zeroth-order solution in the upper Ekman layer is

$$\bar{u}_0 = u_0(x,y) - \frac{\cos \pi y}{2B}\exp(B\bar{\zeta})(\cos B\bar{\zeta} + \sin B\bar{\zeta}), \tag{18}$$

$$\bar{v}_0 = v_0(x,y) + \frac{\cos \pi y}{2B}\exp(B\bar{\zeta})(\cos B\bar{\zeta} - \sin B\bar{\zeta}). \tag{19}$$

The vertical velocity $\overline{W}_0$ may be obtained from integration of (13) with respect to $\bar{\zeta}$ and substitution from (18) and (19), giving

$$B^2\overline{W}_0 = \overline{W}_{01}(x,y) + \beta^*\bar{\zeta}v_0(x,y) + \frac{\beta^*}{2B^2}\cos \pi y \exp(B\bar{\zeta})\cos B\bar{\zeta}$$
$$+\frac{\pi}{2}\sin \pi y \exp(B\bar{\zeta})\cos B\bar{\zeta} - \frac{\beta^*\bar{\zeta}}{4B}\cos \pi y \exp(B\bar{\zeta})(\cos B\bar{\zeta} - \sin B\bar{\zeta}).$$

Rewriting this expression in terms of the outer (interior) variable, we have for large negative $\bar{\zeta}$,

$$B^2\bar{w} = (2E_V)^{\frac{1}{2}}\{\overline{W}_{01} + \beta^* v_0 z (2E_V)^{-\frac{1}{2}}\} + \ldots$$
$$= \beta^* v_0 z + O(E_V^{\frac{1}{2}}).$$

Matching with the interior flow (12) gives $w_0^* \equiv 0$. Finally, the boundary condition (17) at $\bar{\zeta} = 0$ gives

$$\overline{W}_{01} = -\frac{\beta^*}{2B^2}\cos \pi y - \tfrac{1}{2}\pi \sin \pi y. \tag{20}$$

The lower Ekman layer

In this region the stretched co-ordinate is $\tilde{\zeta} = (z+1)(2E_V)^{-\frac{1}{2}}$ and the equations are the same as (13)–(16) with the overbars replaced by tildes, used to denote the lower Ekman layer. Similarly, the solution of (14) and (15) which matches the interior flow for large $\tilde{\zeta}$ is

$$\tilde{u}_0 = u_0(x,y) + \tilde{u}_{02}(x,y)\exp[-(1+i)B\tilde{\zeta}] + \tilde{u}_{03}(x,y)\exp[-(1-i)B\tilde{\zeta}],$$
$$\tilde{v}_0 = v_0(x,y) + \tilde{v}_{02}(x,y)\exp[-(1+i)B\tilde{\zeta}] + \tilde{v}_{03}(x,y)\exp[-(1-i)B\tilde{\zeta}].$$

The boundary condition at $\tilde{\zeta} = 0 (z = -1)$ is $\tilde{u} = \tilde{v} = \tilde{W} = 0$ and hence

$$u_0 + \tilde{u}_{02} + \tilde{u}_{03} = 0, \quad v_0 + \tilde{v}_{02} + \tilde{v}_{03} = 0,$$

which, on substituting in (14), gives

$$\tilde{u}_{02} = i\tilde{v}_{02} = \tfrac{1}{2}(-u_0 - iv_0), \quad \tilde{u}_{03} = -i\tilde{v}_{03} = \tfrac{1}{2}(-u_0 + iv_0).$$

Thus, the zeroth-order solution in the lower Ekman layer is

$$\tilde{u}_0 = u_0 - \exp(-B\tilde{\zeta})(u_0 \cos B\tilde{\zeta} + v_0 \sin B\tilde{\zeta}),$$
$$\tilde{v}_0 = v_0 + \exp(-B\tilde{\zeta})(u_0 \sin B\tilde{\zeta} - v_0 \cos B\tilde{\zeta}).$$

The integral of (13) with respect to $\tilde{\zeta}$ gives

$$B^2 \tilde{W}_0 = \tilde{W}_{02}(x, y) + \beta^* v_0 \tilde{\zeta} + \exp(-B\tilde{\zeta}) \text{ (function of } u_0, v_0),$$

and the undefined function is zero if u_0 and v_0 are zero. Rewriting in terms of the outer variables we have, for large $\tilde{\zeta}$,

$$B^2 \tilde{w} = \beta^* v_0 (z+1) + O(E_V^{\frac{1}{2}}).$$

Recalling that we have already shown that $w_0^* \equiv 0$, matching with the interior flow (12) gives $v_0 \equiv 0$. Now (10a) gives $\partial w_0/\partial z \equiv 0$ and hence the continuity equation (11) gives $\partial u_0/\partial x \equiv 0$. Thus u_0 is a function of y only. We shall show in § 8 that with the imposed wind stress (6), the interior normal velocity must satisfy the boundary condition at the east coast; that is, $u = 0$ at $x = 1$ and $u_0 = 0$ for all x, y.

We have shown that there is no zeroth-order flow in the interior, and using the boundary condition at $\tilde{\zeta} = 0$, we can show that $\tilde{W}_{02} \equiv 0$. Hence there is no zeroth-order flow in the lower Ekman layer. The only zeroth-order flow is in the upper boundary layer, which is a conventional Ekman layer that decreases in thickness with increasing latitude due to the presence of the $(1+\beta^* y)$. The vertical velocity out of the upper layer (Ekman suction) will produce an $O(E_V^{\frac{1}{2}})$ flow in the interior, as we shall see in the next section.

7. Higher-order terms: the interior solution

For the first- and second-order terms the calculation of § 6 is followed with the exception that the boundary conditions at the surface have the simpler form

$$\partial \bar{u}_i/\partial \bar{\zeta}, \quad \partial \bar{v}_i/\partial \bar{\zeta}, \quad \overline{W}_i = 0 \quad (\bar{\zeta} = 0). \tag{21}$$

It is found that all first- and second-order terms are zero, as could have been anticipated as we expect a power series in $E_V^{\frac{1}{2}}$ in the Ekman layers. However, the third-order, $O(E_V^{\frac{1}{2}})$, flow is non-zero because the function $\overline{W}_{01}$ now appears in the matching condition. As for the zeroth-order terms, the solution which matches with the interior flow for large negative $\bar{\zeta}$ is

$$\bar{u}_3 = u_3 + \bar{u}_{32} \exp[(1+i)B\bar{\zeta}] + \bar{u}_{33} \exp[(1-i)B\bar{\zeta}],$$
$$\bar{v}_3 = v_3 + \bar{v}_{32} \exp[(1+i)B\bar{\zeta}] + \bar{v}_{33} \exp[(1-i)B\bar{\zeta}].$$

The boundary conditions (21) at the surface $\bar{\zeta} = 0$, together with the relations obtained by substituting in (14), give

$$\bar{u}_{32} = \bar{u}_{33} = \bar{v}_{32} = \bar{v}_{33} = 0.$$

Thus the flow in the upper Ekman layer is just the zeroth-order Ekman flow plus the $O(E_V^{\frac{1}{2}})$ interior flow. Equation (13) with $i = 3$ gives, on integration with respect to $\bar{\zeta}$,

$$B^2 \overline{W}_3 = \overline{W}_{31}(x, y) + \beta^* v_3 \bar{\zeta},$$

and the boundary condition at $\bar{\zeta} = 0$ implies that $\overline{W}_{31} = 0$. Rewriting in outer variables we have, for large negative $\bar{\zeta}$,

$$B^2 \overline{W}_3 = B^2 (2E_V)^{\frac{1}{2}} (\overline{W}_0 + E_V^{\frac{1}{2}} \overline{W}_1 + \ldots)$$
$$= (2E_V)^{\frac{1}{2}} \overline{W}_{01} + E_V^{\frac{1}{2}} \beta^* v_3 z + O(E_V^{\frac{2}{3}}).$$

The outer (interior) solution, from (12), is

$$B^2 w = E_V^{\frac{1}{2}}(\beta^* v_3 z + w_3^*) + O(E_V^{\frac{3}{4}}), \tag{22}$$

and matching gives

$$w_3^* = \overline{W}_{01}\sqrt{2}. \tag{23}$$

The third-order Ekman layer solution has the same form as the zeroth-order solution as the boundary conditions are unchanged. The appropriate solution is

$$\tilde{u}_3 = u_3 - \exp[-B\tilde{\zeta}](u_3 \cos B\tilde{\zeta} + v_3 \sin B\tilde{\zeta}),$$
$$\tilde{v}_3 = v_3 + \exp[-B\tilde{\zeta}](u_3 \sin B\tilde{\zeta} - v_3 \cos B\tilde{\zeta}),$$
$$B^2\tilde{W}_3 = \tilde{W}_{32}(x,y) + \beta^* v_3 \tilde{\zeta} + \text{exponential terms},$$

where $\tilde{W}_{32}$ may be found from the boundary condition at $\tilde{\zeta} = 0$. Rewriting in outer variables, for large $\tilde{\zeta}$ we have

$$B^2\tilde{w} = E_V^{\frac{1}{2}}\beta^* v_3(z+1) + O(E_V^{\frac{3}{4}}).$$

Matching with the interior solution (22) gives

$$\beta^* v_3 = w_3^* = -\frac{\beta^*}{B^2\sqrt{2}}\cos \pi y - \frac{\pi}{\sqrt{2}}\sin \pi y, \tag{24}$$

using (20) and (23). Now the continuity equation and (10*a*) gives

$$B^2\left(\frac{\partial u_3}{\partial x} + \frac{\partial v_3}{\partial y}\right) = -B^2\frac{\partial w_3}{\partial z} = -\beta^* v_3.$$

Hence

$$B^2\frac{\partial u_3}{\partial x} = \frac{B^2\pi^2}{\beta^*\sqrt{2}}\cos \pi y,$$

and

$$u_3 = \frac{\pi^2 x}{\beta^*\sqrt{2}}\cos \pi y + U_3(y). \tag{25}$$

In §8 we show that the interior normal velocity must satisfy the boundary condition at the east coast. Thus the function $U_3(y)$ is chosen so that $u_3 = 0$ at $x = 1$, and then the interior solution is

$$u = -\left(\frac{E_V}{2}\right)^{\frac{1}{2}}\frac{\pi^2}{\beta^*}(1-x)\cos \pi y + O(E_V), \tag{26}$$

$$v = -\left(\frac{E_V}{2}\right)^{\frac{1}{2}}\left(\frac{\cos \pi y}{B^2} + \frac{\pi}{\beta^*}\sin \pi y\right) + O(E_V), \tag{27}$$

$$w = -\left(\frac{E_V}{2}\right)^{\frac{1}{2}}\frac{\beta^*}{B^2}(z+1)\left(\frac{\cos \pi y}{B^2} + \frac{\pi}{\beta^*}\sin \pi y\right) + O(E_V), \tag{28}$$

$$B^2 = 1 + \beta^* y,$$

where the fact that the fourth- and fifth-order terms can be shown to be zero has been included.

It is seen in (28) that the vertical velocity decreases linearly with depth to zero at the bottom. Thus to this order there is no flow into the lower Ekman layer whose only purpose is to smooth the horizontal velocity to zero. It is clearly of minor importance compared with the upper layer. These general results were obtained by Robinson (1965, chapter 17). Although in the dimensionless variables above, w has the same order of magnitude as the horizontal velocity components, it should be recalled that the scaling for w included the small parameter ϵ. Consequently the actual dimensional vertical velocity is extremely small (a few cm per day using the values in §3). However, the vertical transport over the whole ocean is a significant part of the motion.

The most interesting result, due to the inclusion of the β^* term throughout is the fact that v and w are not zero where the wind stress curl is zero. The interior flow is shifted southward relative to the wind stress field, and can be examined more easily if (27) is rewritten as

$$v = -\left(\frac{E_V}{2}\right)^{\frac{1}{2}}\left(\frac{1}{B^4}+\frac{\pi^2}{\beta^{*2}}\right)^{\frac{1}{2}}\sin\left(\pi y+\tan^{-1}\frac{\beta^*}{B^2\pi}\right)+O(E_V). \tag{29}$$

The phase shift southward is $\tan^{-1}\beta^*/\pi(1+\beta^* y)$, which varies from about $\pi/14$ at $y = 0$ to about $\pi/36$ at $y = 2$. An explanation of this effect is now given. For a uniformly rotating fluid, the Ekman layer suction (or w) is zero where the wind stress curl is zero. On a β-plane with constant east wind stress (that is zero curl) there will be a flow out of the Ekman layer for the northward Ekman transport as the thickness of the layer decreases with increasing latitude. Consequently for a β-plane with variable wind stress, the zeros of v and w in the interior cannot occur at latitudes of zero wind stress curl. Moreover as the transport of the Ekman layer due to the β-effect decreases with increasing latitude whereas the transport associated with the wind stress is periodic, we expect to find that the phase shift decreases with increasing latitude.

Finally, as the u velocity component remains in phase with the wind stress, there will be a north-south asymmetry in the interior flow as will be seen later in figure 2. The structure of the circulation produced by these results will become clear after the flow in the west coast boundary layer has been examined.

8. The east and west coast boundary layers

The interior flow cannot satisfy the conditions of zero velocity at the east and west coasts and boundary layers will be required. As the wind stress was chosen so that there is no east–west transport in the upper Ekman layer, there will not be any upwelling or sinking at the coasts. At the east coast we shall find that the interior normal velocity u is zero and that a weak boundary layer is required to bring the $O(E_V^{\frac{1}{2}})$ tangential flow to zero at $x = 1$. The structure of this boundary layer will be simple exponential decay of the interior flow. This layer would be more interesting if a different wind stress were used which required the layer to remove fluid from or supply fluid to the Ekman layer.

At the west coast the situation is complicated by the need for a substantial north-south return flow to compensate the slow drift in the interior. This return flow of $O(E_H^{\frac{1}{4}})$ will take place in a layer of thickness $E_H^{\frac{1}{4}}$ which also smooths the normal velocity u to zero at $x = 0$. In addition, as on the east coast, a weaker layer is required to bring the $O(E_V^{\frac{1}{2}})$ tangential flow to zero, but will not be considered here as it is not involved in net transport of fluid.

West coast boundary layer

A circumflex will be used to denote variables in the west coast boundary layer and the following new variables are introduced

$$\eta = E_H^{-\frac{1}{3}}x, \quad \hat{U} = E_H^{-\frac{1}{3}}\hat{u}.$$

Expanding in power series as above but this time emphasising the dependence on the small parameter E_H, we have

$$\begin{aligned} \hat{U} &= E_H^{\frac{1}{6}}\hat{U}_1 + E_H^{\frac{1}{3}}\hat{U}_2 + \ldots, \\ \hat{v} &= E_H^{\frac{1}{6}}\hat{v}_1 + E_H^{\frac{1}{3}}\hat{v}_2 + \ldots, \\ \hat{w} &= E_H^{\frac{1}{2}}\hat{w}_1 + \ldots, \\ \hat{P} &= E_H^{\frac{1}{2}}\hat{P}_1 + \ldots. \end{aligned}$$

As the horizontal velocities in the interior are independent of z and as there is no flow from the Ekman layer into this layer, the vertical velocity in this layer must be the same order as in the interior. Substitution into the linear terms of (2)–(5) and retention of the lowest order terms gives

$$\begin{aligned} (1+\beta^* y)\hat{v}_1 &= \hat{P}_{1\eta}, \\ (1+\beta^* y)\hat{U}_1 &= -\hat{P}_{1y} + \hat{v}_{1\eta\eta}, \\ 0 &= -\hat{P}_{1z}, \\ \hat{U}_{1\eta} + \hat{v}_{1y} &= 0. \end{aligned} \tag{30}$$

Elimination of $\hat{P}_1$ and $\hat{U}_1$ gives the following equations,

$$\hat{v}_{1\eta\eta\eta} - \beta^* \hat{v}_1 = 0, \quad \hat{v}_{1z} = 0, \tag{31}$$

and the solution for $\hat{v}_1$, which tends to zero as $\eta \to \infty$ is

$$\hat{v}_1 = \hat{v}_{11}(y)\exp(\omega\beta^{*\frac{1}{3}}\eta) + \hat{v}_{12}(y)\exp(\omega^2\beta^{*\frac{1}{3}}\eta), \tag{32}$$

where $\omega = \frac{1}{2}(-1 + i\sqrt{3})$. The boundary condition at $\eta = 0$ gives

$$\hat{v}_{11} + \hat{v}_{12} = 0.$$

Proceeding now to the variable $\hat{U}_1$ which has to match with the interior flow we have, from the continuity equation (30), that

$$\hat{U}_{1\eta} = -\frac{d\hat{v}_{11}}{dy}\{\exp(\omega\beta^{*\frac{1}{3}}\eta) - \exp(\omega^2\beta^{*\frac{1}{3}}\eta)\},$$

and thus
$$\hat{U}_1 = \hat{U}_{11}(y) - \frac{1}{\omega^2\beta^{*\frac{1}{3}}}\frac{d\hat{v}_{11}}{dy}\{\omega\exp(\omega\beta^{*\frac{1}{3}}\eta) - \exp(\omega^2\beta^{*\frac{1}{3}}\eta)\}. \tag{33}$$

So for large η,
$$\begin{aligned} \hat{u} &= E_H^{\frac{1}{2}}\hat{U}_1 + \ldots \\ &= E_H^{\frac{1}{2}}\hat{U}_{11}(y) + O(E_H^{\frac{2}{3}}). \end{aligned}$$

The interior solution, written in terms of inner (boundary layer) variables, becomes

$$\begin{aligned} u &= -\left(\frac{E_V}{2}\right)^{\frac{1}{2}}\frac{\pi^2}{\beta^*}(1 - E_H^{\frac{1}{3}}\eta)\cos\pi y + \ldots \\ &= -E_H^{\frac{1}{2}}\left(\frac{E_V}{2E_H}\right)^{\frac{1}{2}}\frac{\pi^2}{\beta^*}\cos\pi y + O(E_H^{\frac{2}{3}}). \end{aligned}$$

Therefore matching gives

$$\hat{U}_{11}(y) = -\left(\frac{E_V}{2E_H}\right)^{\frac{1}{2}} \frac{\pi^2}{\beta^*} \cos \pi y. \tag{34}$$

The boundary condition on $\hat{U}_1$ at $\eta = 0$ is, from (33),

$$O = \hat{U}_{11} - \frac{d\hat{v}_{11}}{dy} \frac{(\omega - 1)}{\omega^2 \beta^{*\frac{1}{3}}},$$

which implies that

$$\frac{d\hat{v}_{11}}{dy} = \frac{\beta^{*\frac{1}{3}}}{\omega^2 - \omega} \hat{U}_{11} = -\left(\frac{E_V}{2E_H}\right)^{\frac{1}{2}} \frac{i\pi^2}{\beta^{*\frac{2}{3}}\sqrt{3}} \cos \pi y$$

and

$$\hat{v}_{11} = -\left(\frac{E_V}{2E_H}\right)^{\frac{1}{2}} \frac{i\pi}{\beta^{*\frac{2}{3}}\sqrt{3}} \sin \pi y + i\hat{v}_{13},$$

where $\hat{v}_{13}$ is a constant. Thus, from (32)–(34),

$$\hat{v}_1 = \left\{\left(\frac{E_V}{2E_H}\right)^{\frac{1}{2}} \frac{2\pi}{\beta^{*\frac{2}{3}}\sqrt{3}} \sin \pi y - 2\hat{v}_{13}\right\} \exp\left(-\tfrac{1}{2}\beta^{*\frac{1}{3}}\eta\right) \sin \tfrac{1}{2}\sqrt{(3)}\beta^{*\frac{1}{3}}\eta,$$

$$\hat{u} = -\left(\tfrac{1}{2}E_V\right)^{\frac{1}{2}} \frac{\pi^2}{\beta^*} \cos \pi y \left\{1 - \exp\left(-\tfrac{1}{2}\beta^{*\frac{1}{3}}\eta\right)\left(\frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} \beta^{*\frac{1}{3}}\eta - \cos \frac{\sqrt{3}}{2} \beta^{*\frac{1}{3}}\eta\right)\right\}, \tag{35}$$

which brings the interior east–west flow smoothly to zero at the west coast. The remaining constant is determined by ensuring that the volume flux into the boundary layer equals the return flux in the layer. As the model has unit depth, the volume flux into the boundary layer between the latitudes $y = 0$ and $y = \frac{1}{2}$ is

$$\int_{y=0}^{\frac{1}{2}} -u(0, y, z)\,dy = \left(\tfrac{1}{2}E_V\right)^{\frac{1}{2}} \frac{\pi^2}{\beta^*} \int_0^{\frac{1}{2}} \cos \pi y\,dy = \left(\tfrac{1}{2}E_V\right)^{\frac{1}{2}} \frac{\pi}{\beta^*}.$$

The northward volume flux in the boundary layer past the latitude $y = \frac{1}{2}$ is

$$\begin{aligned}
\int_{x=0}^{\delta} \hat{v}(x, \tfrac{1}{2}, z)\,dx &= \int_{\eta=0}^{\infty} E_H^{\frac{1}{3}} \hat{v}(\eta, \tfrac{1}{2}, z)\,d\eta \\
&= E_H^{\frac{1}{3}} \left\{\left(\frac{E_V}{2E_H}\right)^{\frac{1}{2}} \frac{2\pi}{\beta^{*\frac{2}{3}}\sqrt{3}} - 2\hat{v}_{13}\right\} \int_0^{\infty} \exp\left(-\tfrac{1}{2}\beta^{*\frac{1}{3}}\eta\right) \sin \tfrac{1}{2}\sqrt{(3)}\beta^{*\frac{1}{3}}\eta\,d\eta \\
&= \left(\frac{E_V}{2}\right)^{\frac{1}{2}} \frac{\pi}{\beta^*} - \frac{E_H^{\frac{1}{3}}\sqrt{3}}{\beta^{*\frac{1}{3}}} \hat{v}_{13}.
\end{aligned} \tag{36}$$

Hence continuity is only satisfied if $\hat{v}_{13} = 0$, and the return flow along the west coast is given by

$$\hat{v} = E_H^{\frac{1}{3}} \left(\frac{E_V}{2E_H}\right)^{\frac{1}{2}} \frac{2\pi}{\beta^{*\frac{2}{3}}\sqrt{3}} \sin \pi y \exp\left(-\tfrac{1}{2}\beta^{*\frac{1}{3}}\eta\right) \sin \tfrac{1}{2}\sqrt{(3)}\beta^{*\frac{1}{3}}\eta. \tag{37}$$

East coast boundary layer

Turning now to the east coast boundary layer, the appropriate stretched co-ordinate is $\bar{\eta} = (1 - x)E_H^{-\frac{1}{3}}$, and the equations for $\hat{v}_1$ corresponding to (31) are

$$\hat{v}_{1\bar{\eta}\bar{\eta}\bar{\eta}} + \beta^* \hat{v}_1 = 0, \quad \hat{v}_{1z} = 0.$$

The solution that is bounded as $\bar{\eta} \to \infty$ is

$$\hat{v}_1 = \hat{v}_E(y) \exp(-\beta^{*\frac{1}{3}}\bar{\eta})$$

and the continuity equation gives

$$\hat{U}_1 = \hat{U}_E(y) - \frac{d\hat{v}_E}{dy}\frac{1}{\beta^{*\frac{1}{3}}}\exp(-\beta^{*\frac{1}{3}}\bar{\eta}).$$

So for large $\bar{\eta}$,

$$\hat{u} = E_H^{\frac{1}{3}}\hat{U}_1 + \ldots = E_H^{\frac{1}{3}}\hat{U}_E + O(E_H^{\frac{2}{3}}).$$

The interior solution (25), written in terms of the inner variable, is given by

$$u = E_V^{\frac{1}{2}}\left\{\frac{\pi^2(1-E_H^{\frac{1}{3}}\bar{\eta})}{\beta^*\sqrt{2}}\cos\pi y + U_3(y)\right\} + \ldots$$

$$= E_V^{\frac{1}{2}}\left\{\frac{\pi^2\cos\pi y}{\beta^*\sqrt{2}} + U_3(y)\right\} + O(E_V^{\frac{5}{6}}).$$

Therefore matching gives

$$\hat{U}_E = \left(\frac{E_V}{E_H}\right)^{\frac{1}{2}}\left\{\frac{\pi^2\cos\pi y}{\beta^*\sqrt{2}} + U_3(y)\right\}$$

and the boundary condition on $\bar{\eta} = 0$ for u and v gives

$$\hat{v}_E = 0, \quad \hat{U}_E = d\hat{v}_E/dy = 0.$$

Therefore

$$U_3(y) = -\frac{\pi^2\cos\pi y}{\beta^*\sqrt{2}}$$

showing that the interior velocity u must satisfy the east coast boundary condition.

9. Discussion

The west coast solution obtained in (35) and (37) closely resembles the integrated solution given by Munk (1950) and includes an offshore counter current with transport of magnitude 17 % of the main current. There is no phase shift in this current and therefore it does not form, with the interior flow, the simple closed gyres that Munk obtained. As shown in figure 2, some fluid from the west coast current in the southern half-basin moves out of the southern gyre and enters the northern gyre, thereby producing a flow from south-west to north-east. There is an apparent lack of continuity until it is realized that the northern gyre is a region of upflow into the Ekman layer with a corresponding downflow out of the Ekman layer into the southern gyre. It remains to verify that the transport from the south-west to the north-east is indeed returned by the Ekman transport in the upper layer. The flow out of the region $0 \leqslant y \leqslant 1$ into the interior of adjacent regions is, from (27),

$$\int_{x=0}^{1} -v(x,0,z)\,dx + \int_{x=0}^{1} v(x,1,z)\,dx = \left(\frac{E_V}{2}\right)^{\frac{1}{2}}\left(1+\frac{1}{1+\beta^*}\right). \tag{38}$$

The downflow out of the Ekman layer in the region $0 \leqslant y \leqslant 1$ is, by (28),

$$\int_{y=0}^{1}\int_{x=0}^{1} -w(x,y,0)\,dx\,dy = \left(\frac{E_V}{2}\right)^{\frac{1}{2}}\int_0^1 \frac{\beta^*}{1+\beta^* y}\left\{\frac{\cos\pi y}{1+\beta^* y}+\frac{\pi\sin\pi y}{\beta^*}\right\}dy$$

$$= \left(\frac{E_V}{2}\right)^{\frac{1}{2}}\left[-\frac{\cos\pi y}{1+\beta^* y}\right]_0^1 = \left(\frac{E_V}{2}\right)^{\frac{1}{2}}\left(1+\frac{1}{1+\beta^*}\right),$$

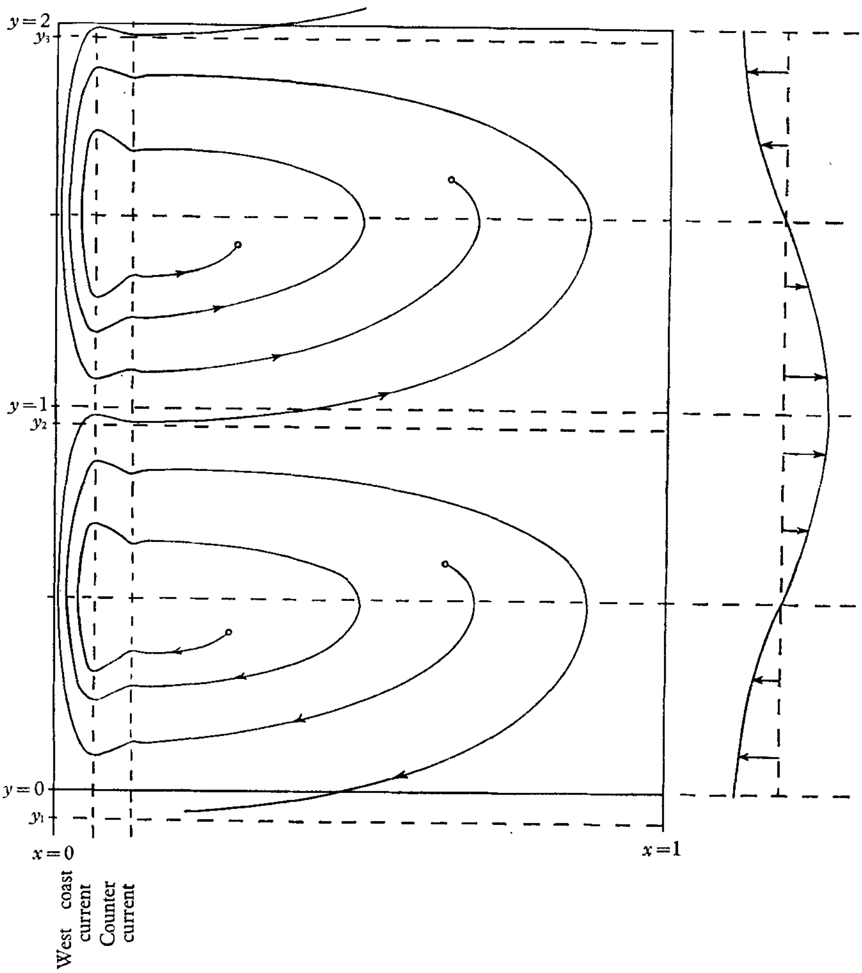

FIGURE 2. Plan view of the interior flow. The velocity components v, w are zero on the lines $y = y_1, y_2, y_3$. On the right is shown the surface stress distribution. Downflow out of the Ekman layer occurs in the region $y_1 < y < y_2$. Upflow occurs in the region $y_2 < y < y_3$.

thus satisfying continuity. Although the phase shift at latitude $y = 1$ is small, as the wind stress has a maximum there, the actual transport north-eastwards is appreciable. The fraction of the west coast current that moves north-eastwards is, from (36) and the second term of (38),

$$(1+\beta^*)^{-1}/\pi\beta^{*-1},$$

which corresponds to about 13 %, using the value $\beta^* = 0{\cdot}71$.

Clearly in the standard method of integrating vertically and dealing with volume transports, these results could not be foreseen as there is no net transport across the latitudes of zero wind stress curl. Thus the variation of the Coriolis parameter with latitude provides a mechanism for producing a drift of fluid north-eastwards across the ocean, a feature of the North Atlantic circulation.

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